Fourier Transform and Finite Difference Solutions of MHD Flow and Heat Transfer in A Square Duct with Joule Dissipation

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***Abstract*—This study presents the effect of magnetic field on the fully developed Newtonian flow and Entrance heat transfer in a square duct. Double Fourier transform is used to introduce closed forms for velocity, temperature, friction and Nusselt number. The no slip condition and *H*1 thermal condition with four heated walls are considered. The present analytical solution is compared with special available results. The present analytical results are verified, in all cases, by introducing a finite difference method. The effects of Hartmann number and Brinkman number on the velocity, temperature, the friction factor and Nusselt number are investigated. New results are introduced for relatively high values of Brinkman number.**

***Index Terms*— MHD flow, Double Fourier series, finite difference method, square duct, friction factor, Joule dissipation, Entrance heat transfer, Nusselt number.**

I. INTRODUCTION

EXACT solutions are very important for checking the accuracies of numerical and empirical methods [1]. Aklyidiz and Jones studied the generation of the steady flow of an Oldroyd B fluid in a rectangular duct. They used Laplace transform to separate out the time dependence from the governing equations and double Fourier transform was used to solve the resulting Poisson problem in the space variables. Pathak [3] presented a discussion of the flow of dusty viscous fluid through hexagonal duct.

The Developed integral transform is employed to solve the problem. The laminar viscous flow through rectilinear duct has been studied by Fan [4]. Hunt [5] introduced an exact solution for laminar motion of a conducting liquid in a rectangular duct. He solved the coupled equations of velocity and magnetic field. Morini, et al. [6] studied the effect of viscous dissipation on the temperature distribution through rectangular ducts. The flow was assumed to be laminar, fully developed with constant properties. They calculated the velocity and temperature using finite Fourier series. Morini, [7], introduced an analytical solution of temperature Nusselt numbers in rectangular ducts with constant axial heat flux. He studied the effects of the eight versions of H1 condition on the temperature distributions and Nusselt number. Ewis [8] introduces a finite difference method solution for non Newtonian fully developed fluid flow and developing heat transfer in a rectangular duct filled with porous medium. He studied the effect of porosity and dissipation on flow and temperature numerically. Hooman et al. [9] investigated analytically the First and the Second Law (of Thermodynamics) characteristics of fully developed forced convection inside a porous-saturated duct of rectangular cross-section. The Darcy-Brinkman flow model is employed. Three different types of thermal boundary conditions are examined. Expressions are presented for the Nusselt number, the Bejan number, and the dimensionless entropy generation rate in terms of the system parameters.

In the present paper, the effects of Hartmann number and Brinkman number on the velocity, friction factor, temperature and Nusselt number through a square duct are studied. The double Fourier transform is used to solve, exactly, the governing equations and obtaining closed forms for the velocity, friction factor temperature, and Nusselt number. The H1 thermal boundary condition with peripherally and constantly heated walls and axially constant rate of temperature are considered. The finite difference solution of governing equations is introduced to verify the convergence of Fourier series. New results are obtained and discussed at relatively high Brinkman number. The present results are compared with available analytical and numerical results in special cases.

II. FORMULATION OF THE PROBLEM

The flow is assumed to be steady, conducting, laminar and fully developed and is driven by a constant pressure gradient *dp*/*dx*. The no-slip condition is applied at the walls of square duct. The duct width is assumed to be 2a and as shown in Fig. 1. The flow is subjected to laterally normal magnetic field with lo magnetic Reynolds number. So, the induced magnetic field is neglected. The effects Joule dissipation and magnetic field are taken into consideration.

Under the above assumptions, the momentum and continuity equations take the form.

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{\sigma B_0^2}{k}u=\frac{1}{k}\frac{dp}{dx} \quad (1)$$

$$\int_0^{2a}\int_0^{2a} u\, dy\, dz = 4a^2 u_m \quad (2)$$

with the following no slip condition,

$$u(0,z) = u(y,0) = u(2a,z) = u(y,2a) = 0, \quad (3)$$

where,
k is the fluid viscosity,
k_p is the porous medium shape parameter.
u_m is the average of velocity over the cross-section of the duct.

Dimensionless variables are introduced as:

$$, \ Y = \frac{y}{D_h}, \ Z = \frac{z}{D_h}, \ U = \frac{u}{u_m},$$

$$R_e = \frac{\rho\, u_m\, D_h}{k}, \text{ (Reynolds Number)}$$

$$f = \frac{(-dp/dx)\, D_h}{2\rho\, u_m^2} \text{ (friction factor),}$$

$$H_a^2 = \frac{\sigma B_0^2\, D_h^2}{k} \text{ ,(Hartmann number)}$$

$$D_h = 4A/p' \text{ (Hydraulic radius =4*area/witted perimeter)}$$

Then, Eqs. (1-3) are written in dimensionless form as

$$\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} - H_a^2\, U + 2 f \,\mathrm{Re} = 0 \quad (4)$$

$$\int_0^1 \int_0^1 U\, dY\, dZ = 1 \quad (5)$$

$$U(0,Z) = U(Y,0) = U(1,Z) = U(Y,1) = 0, \quad (6)$$

Under the above assumptions and neglecting axial thermal conduction, natural convection, energy sources, with rigid and non porous duct walls, the energy equation may be written as [7]

$$\frac{1}{\alpha} u \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\sigma B_0^2}{K} u^2 \quad (7)$$

where, α is fluid thermal diffusivity and K is the thermal conductivity.

The $H1$ boundary condition states that the temperature at duct walls is uniform on the heated length of square perimeter and increases linearly with the x longitudinal coordinate [10]. Thus, the thermal boundary conditions are written as.

$$T(0,z) = T(y,0) = T(2a,z) = T(y,2a) = T_w \quad (8)$$

According to $H1$ boundary condition, the following relations are satisfied [7].

$$\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x} = q' \Big/ \rho D_h^2 u_m c_p \quad (9a)$$

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad (9b)$$

where, T_b is the bulk mean temperature distribution of the cross-section of duct, c_p is the fluid specific heat and q' is the thermal power per unit of length.

In this case, the following dimensionless quantities are introduced,

$$\theta = \frac{K(T - T_w)}{q'} \text{ (dimensionless temperature)}$$

$$B_r = k\, u_m^2 \big/ q' \text{ , (Brinkman number)}$$

Then, the dimensionless forms of energy equation and its boundary conditions are written as

$$\frac{\partial^2 \theta}{\partial Y^2} + \frac{\partial^2 \theta}{\partial Z^2} - U + B_r H_a^2\, U^2 = 0 \quad (10)$$

$$\theta(0,Z) = \theta(Y,0) = \theta(1,Z) = \theta(Y,1) = 0 \quad (11)$$

III. ANALYTICAL SOLUTION

As with the usual solution of differential equations, we assume a solution that satisfies the boundary equations. Trigonometric series can generally be used to satisfy many types of boundary conditions. For our problem,

$$U(Y,Z) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin(n\pi Y)\sin(m\pi Z), \quad (12)$$

where, A_{nm} is the double Fourier sine transform of the two dimensions function and is given by

$$A_{nm} = \int_0^1\int_0^1 U(Y,Z)\sin(n\pi Y)\sin(m\pi Z)\, dY dZ \quad (13)$$

The solution of the problem (4-6) will be obtained by means of the double Fourier sine transform. Taking double Fourier sine transform (DFST) of both sides of Eq.(4), we find that

$$A_{nm} = \begin{cases} \dfrac{8 f \,\mathrm{Re}/\pi^2}{nm\left[\pi^2\left(n^2 + m^2\right) + H_a^2\right]} & n \text{ and } m \text{ are odd numbers} \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

$$\left(f\,\mathrm{Re}\right)^{-1} = \frac{\pi^6}{128} \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{n^2 m^2\left[\left(n^2 + m^2\right) + H_a^2/\pi^2\right]} \quad (15)$$

$$U(Y,Z) = \frac{\pi^2}{4} \frac{\sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \left\{nm\left[\left(n^2+m^2\right) + H_a^2/\pi^2\right]\right\}^{-1}}{\sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \left\{n^2 m^2\left[\left(n^2+m^2\right) + H_a^2/\pi^2\right]\right\}^{-1}} \sin(n\pi Y)\sin(m\pi Z)$$

(16)

The central (maximum) velocity is one of the important values and may be calculated by substituting in Eq. (13) at (Y, Z)=(1/2, 1/2). So the closed form the central velocity U_c is written as

$$U_c = \frac{\pi^2}{4}\frac{\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\left\{nm\left[\left(n^2+m^2\right)+H_a^2/\pi^2\right]\right\}^{-1}}{\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\left\{n^2m^2\left[\left(n^2+m^2\right)+H_a^2/\pi^2\right]\right\}^{-1}}(-1)^{(n+m-2)/2} \quad (17)$$

In a similar manner, the Fourier sine transform is applied on Eq. (10) taking the thermal boundary condition (11) into consideration. Thus we can introduce another closed form for temperature as

$$\theta(Y,Z) = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(n\pi Y)\sin(m\pi Z) \quad (18)$$

where,

$$B_{nm} = \int_0^1\int_0^1 \theta(Y,Z)\sin(n\pi Y)\sin(m\pi Z)\,dY\,dZ \quad (19)$$

is the double Fourier sine transform of temperature.

Taking the double Fourier sine transform of both sides on Eq. (10), we may write

$$\left[\pi^2\left(n^2+m^2\right)\right]B_{nm} = -A_{nm} + B_r\int_0^1\int_0^1 U^2\sin(n\pi Y)\sin(m\pi Z)\,dY\,dZ \quad (20)$$

The last double integration may be calculated after importing the double Fourier expansion of the Joule dissipation term. Thus,

$$F_{nm}\left[\left(\frac{\partial U}{\partial Y}\right)^2+\left(\frac{\partial U}{\partial Z}\right)^2\right] = \int_0^1\int_0^1 U^2\sin(n\pi Y)\sin(m\pi Z)\,dY\,dZ$$

$$= \sum_{i=1,3}^{\infty}\sum_{j=1,3}^{\infty}\sum_{k=1,3}^{\infty}\sum_{k=1,3}^{\infty}A_{ij}A_{kl}\int_0^1\int_0^1 \sin(i\pi Y)\sin(j\pi Z)\sin(k\pi Y)\sin(l\pi Z)\sin(n\pi Y)\sin(m\pi Z)\,dY\,dZ \quad (21)$$

where,

F_{nm} is the double Fourier sine transform for Joule dissipation term.

Thus, the temperature transform B_{nm} is rewritten as

$$B_{nm} = \frac{B_rC_{nm}-A_{nm}}{\pi^2\left(n^2+m^2\right)} \quad (22)$$

The temperature can be obtained exactly using the inverse Fourier transform (20) as

$$\theta(Y,Z) = 4\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{B_rC_{nm}-A_{nm}}{\pi^2\left(n^2+m^2\right)}\sin(n\pi Y)\sin(m\pi Z) \quad (23)$$

Finally, the Nusselt number can be obtained by an energy balance on the heated perimeter of square duct: it is expressed as [7]

$$N_{u,H1} = -\frac{1}{4\,\theta_b} \quad (24)$$

where, θ_b is the dimensionless bulk temperature, which is given in the form [8]

$$\theta_b = \int_0^1\int_0^1 U(Y,Z)\theta(Y,Z)\,dY\,dZ \quad (25)$$

Using the inverse Fourier transform, the dimensionless bulk temperature is written as

$$\theta_b = \int_0^1\int_0^1 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin(n\pi Y)\sin(m\pi Z)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}\sin(n\pi Y)\sin(m\pi Z)\,dY\,dZ$$

$$=16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_{nm}B_{kl}\int_0^1\int_0^1 \sin(n\pi Y)\sin(m\pi Z)\sin(k\pi Y)\sin(l\pi Z)\,dY\,dZ \quad (26)$$

After many mathematical steps, we find that,

$$\theta_b = 4\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}A_{nm}B_{nm} \quad (27)$$

IV. NUMERICAL SOLUTION

The numerical solution of coupled Eqs. (4-5) with boundary conditions (6) is characterized by the replacement of continuous derivatives by finite difference representations [8]. These differences are defined at **$(M+1)^2$** nodes of the two-dimensional square mesh. The Crank-Nicolson finite difference method is used for the second derivatives in Eq. (4). Applying Simpson's 1/3 rule for double integral, the continuity Eq. (6) is transformed to a linear algebraic equation [12]. The linear coupled system may be decoupled using Keller method [13] and solved using successive over relaxation (SOR) to reduce the huge storage matrices in computer [8]. Another finite-difference form of the energy equation (10) and a Simpson's 1/3 scheme for equation (25) are presented to compute the temperature and Nusselt number [8]. The iterative SOR method is used of accuracy up to $\left|u_c^{new}-u_c^{old}\right| \le \varepsilon$ and $\left|\theta_c^{new}-\theta_c^{old}\right| \le \varepsilon$, where, $\varepsilon \le 10^{-12}$. The stability and convergence of numerical solutions are functions of the relaxation parameter, the accuracy and the mesh sizes. The truncation errors of the difference schemes are $O(\Delta^2 = 1/M^2)$. Thus the difference equations tend to the differential equations as Δ^2 tends to zero. Another measure of stability and convergence is the linear relation between any of mesh size Δ^2 and the variable U or θ [8].

V. RESULTS AND DISCUSSION

By a simple code (using MATLAB) containing nested loops, our results can be tabulated and drawn of high accuracy and resolution.

Table 1 shows the required number of terms (N^2) to obtain the required accuracy of our results (U_c = U(0.5, 0.5) and $\theta_c = \theta(0.5, 0.5)$. It is observed that the accuracies between 10^{-4} and 10^{-8} are obtained from computations of number of terms between 10^2 and 1000^2, respectively. It is observed that the present results are of good agreement with the available analytical results [6, 11].

Table 2 shows the effect of the porosity parameter s on the maximum velocity U_c, minimum Temperature θ_c and friction factor fRe (Br=0, s = 5, 10, 15, 20, N=1000). It is observed that increasing s decreases the maximum velocity U_c and increases the friction factor because the porous medium resists the motion and generates a high friction between the two phases of fluid field and porous medium. It is also observed that increasing s increases the minimum temperature, because of cooling the fluid.

Table 3 shows the effects of the porosity parameter s and Brinkman number on the central Temperature θ_c (Br=0.01, 0.0176, 0.0352, s =0, 5, 10, 15, 20, N=50). It is observed that increasing B_r increases the minimum temperature, because of the dissipation of due to higher viscosity. This increasing is more pronounced at higher porosity parameter *s.* The present results are of good agreement with the available analytical results [6] with s = 0.

Fig. 2 shows the effect of porosity s on the velocity at the central plane of duct cross-section. It is observed that increasing s decreases the maximum velocity because the porous medium resists the motion. The present results are of good agreement with the available numerical results [8]. The effect of Brinkman number B_r on temperature profile at the central plane of duct is illustrated in Fig. 3 with s=0. It is observed that increasing B_r increases temperature due to increasing the viscous dissipation. The present results are of good agreement with the analytical results [6]. Fig. 4 shows that decreasing s decreases the temperature, because of cooling the fluid. Fig. 5 shows the effect of porosity s on the temperature profile at the central plane of duct cross-section for Br=0.1. It is observed that increasing s decreases the central temperature because of high velocity at the center of duct. It is also observed increasing s decreases the temperature derivative near the duct walls due to the high derivative of velocity at walls which is more pronounced for relatively high Brinkman number.

VI. CONCLUSIONS

The double Fourier transform is applied on the governing equations of conducting fluid flow and heat transfer to introduce exact solutions for the velocity and temperature in a square duct. The *H1* thermal condition is studied. It is observed that the magnetic force represents a resistance force which decreases the velocity profile but it increases the temperature profile because of cooling the fluid. This study is useful when numerical and experimental solutions of this problem need verifications of stability and convergence. The effect of porosity parameter s on the temperature derivative near the duct walls is more pronounced for relatively high Brinkman number due to the high derivative of velocity at walls.

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Table 1.Generation of convergence of the maximum velocity U_c, friction factor *fRe* (*Ha*=0, 5, 10, *Br*=0).

Ha	U_c			*fRe*		
	DFS(*N*=60)	DFS(*N*=100)	FDM	DFS(*N*=60)	DFS(*N*=60)	FDM
0 Comparison	2.0962466	2.0962540 2.0962 [11]	2.0962609 2.0963 [8]	1.4227128	1.4227088 1.42271[11]	1.4227390 1.422741[15]
5	1.8320644	1.8320820	1.8321074	30.832726	30.832539	30.833630
10	1.4973070	1.4973558	1.4974321	76.811688	76.812846	76.815656

Table 2. Generation of convergence of the central temperature θ_c, Nusselt number $N_{u,H1}$ (*Ha*=0, *Br*=0, *M*=200).

Ha	$-\theta_c$			$N_{u,H1}$		
	DFS(*N*=60)	DFS(*N*=100)	FDM	DFS(*N*=60)	DFS(*N*=60)	FDM
0 Comparison	1.1559122	1.1559090 1.156 [6]	1.1559302	3.6079250	3.6079452 3.608 [7]	3.6078653
5	1.0843505	1.0843439	1.0843655	3.0832726	3.0832539	3.0833630
10	.09820414	.09820266	.09820572	4.4855926	4.4857278	4.4855114

Table 3. Effect of the Hartmann number *Ha* Brinkman number B_r on the central temperature θ_c (*N*=100, *M*=200)

B_r	0. 0.01		0.0176		0.0352	
Ha	DFS	FDM	DFS	FDM	DFS	FDM
0	-1.15590897	-1.15593024	-1.15590897	-1.15593024	-1.15590897	-1.15593024
5	-.066179003	-.066179804	-.0340649051	-.034064680	.0403045836	.040307185
10	.0363067813	.0363115189	0.1385339600	0.138544617	0.375270585	0.375294952

Table 4. Effect of the Hartmann number *Ha* Brinkman number B_r on the Nusselt number $N_{u,H1}$ (*N*=100, *M*=200)

B_r	0. 0.01		0.05		0.1	
Ha	DFS	FDM	DFS	FDM	DFS	FDM
0	3.6079452	3.6078653	3.6079452	3.6078653	3.6079452	3.6078653
5	6.2177782	6.2176230	-4.6786961	-4.6785187	-1.4664054	-1.4663553
10	-14.078968	-14.077212	-80.201676	-80.196060	-36.810140	-36.807628

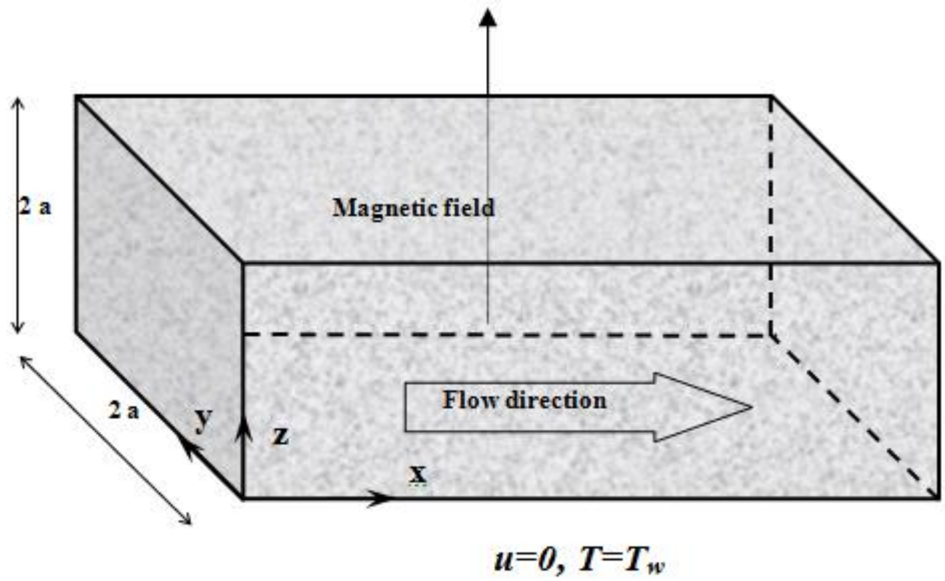

Fig.1. Duct Configuration and boundary conditions

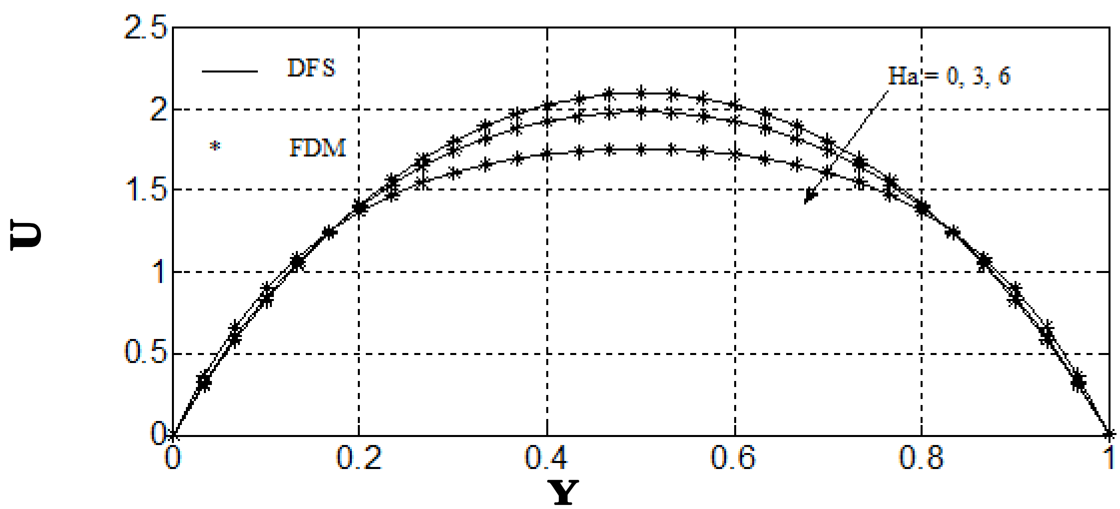

Fig. 2. Effect of Hartmann number Ha on the central plane velocity profile U(Y, 0.5)

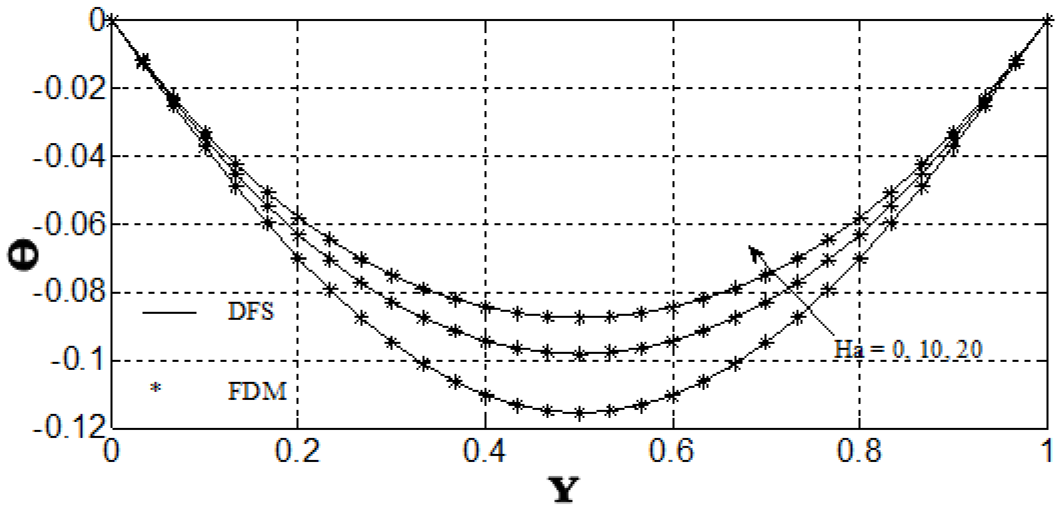

Fig. 3. Effect of Hartmann number H_a on the central plane temperature profile θ (Y, 0.5) with B_r=0.

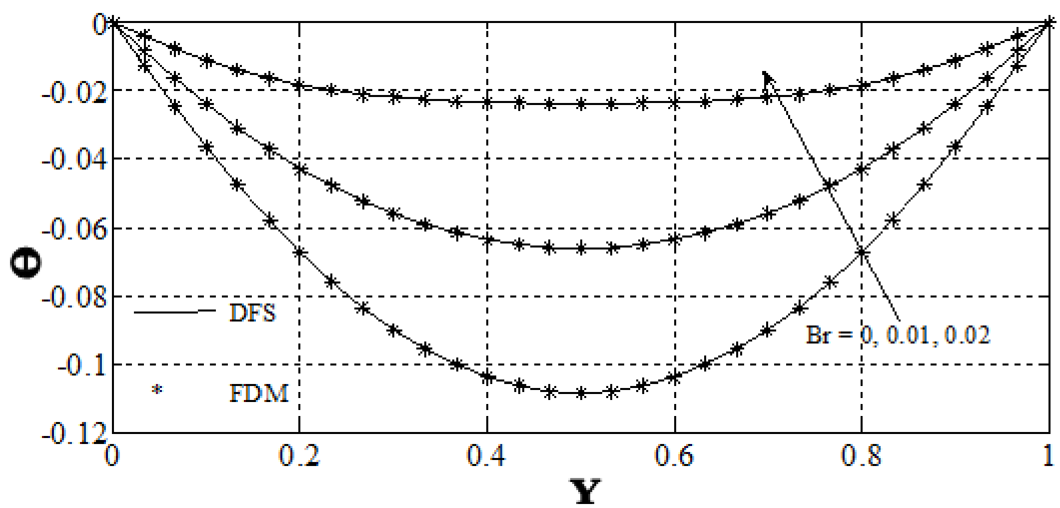

Fig. 4. Effect of the Brinkman number B_r on the central plane temperature profile θ (Y, 0.5) with H_a=5.

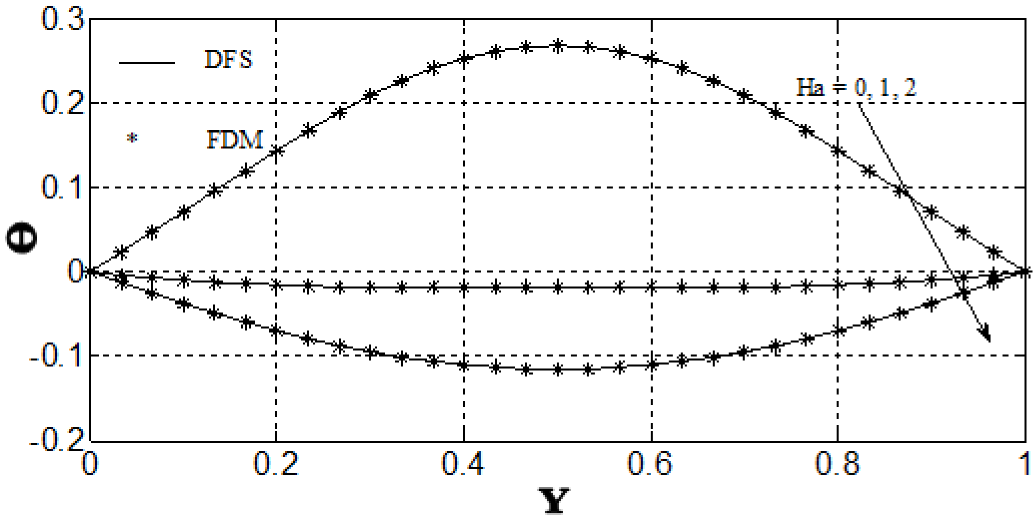

Fig. 5. Effect of Hartmann number H_a on the central plane temperature profile θ (Y, 0.5) with Br=0.5.