The Continuity and Differentiation of Complex Fuzzy Functions for New Fuzzy Quantities

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Abstract—In this paper, the continuity of complex fuzzy functions mapping generalized rectangular valued bounded closed complex complement normalized fuzzy numbers into itself are studied. Some important theorems of fuzzy derivatives of bounded closed complex complement normalized fuzzy number valued functions are proved.

Keywords—Complex Fuzzy numbers, Fuzzy Continuity, Fuzzy Derivatives.

1. Introduction

The concepts of fuzzy sets and fuzzy numbers were first introduced by Zadeh [26, 27] in 1965 and 1975. Many authors have interested in the study of the theory of fuzzy numbers (see [12, 21, 23]). It is well known that fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley [1, 2, 3] in 1989 – 1992. He gave some elementary properties of fuzzy complex numbers and fuzzy complex analysis, Buckley in [3] suggested that introducing a metric on the space of fuzzy complex numbers provide to study convergence, continuity and differentiation of fuzzy complex function (see [4, 7-8, 10-11, 13-16, 22, 25, 28]). Several scholars have extensively studied the theory of fuzzy complex numbers and fuzzy complex analysis (see [5, 6, 9,17-20, 29]).

In section two, we first review the definitions and basic properties related to fuzzy sets. We will also present the notations needed in the rest of the paper. In the next section, we give our definitions of the complement normalized fuzzy numbers (CNFNs), bounded closed complex CNFNs (BCCCNFNs), generalized rectangular valued BCCCNFNs (GRVBCCCNFNs) and discuss some of their basic properties. In the fourth section, some theorems of fuzzy continuity of such functions are proved. The last section contains main results and conclusions related to the fuzzy derivative of complex fuzzy functions.

2. Preliminaries

A fuzzy set \( \tilde{A} \) defined on the universal set \( X \) is a function \( \mu(\tilde{A}, x) : X \rightarrow [0,1] \). Frequently, we will write \( \mu(x) \) instead of \( \mu(\tilde{A}, x) \). The family of all fuzzy sets in \( X \) is denoted by \( \mathcal{F}(X) \). The strong \( \alpha \)-level of a fuzzy set \( \tilde{A} \), denoted by \( \alpha^{+}\tilde{A} \), is the non-fuzzy set of all elements of the universal set that belongs to the fuzzy set \( \tilde{A} \) at least to the degree \( \alpha \in [0,1] \). The weak \( \alpha \)-level \( \alpha^{\tilde{A}} \) of a fuzzy set \( \tilde{A} \in \mathcal{F}(X) \) is the crisp set that contains all elements of the universal set whose membership grades in the given set are greater than \( \alpha \) but do not include the specified value of \( \alpha \). The largest value of \( \alpha \) for which the \( \alpha \)-level is not empty is called the height of a fuzzy set \( \tilde{A} \) denoted \( \alpha^{\tilde{A}}_{\text{max}} \). The core of a fuzzy set \( \tilde{A} \) is the non-fuzzy set of all points in the universal set \( X \) at which \( \sup_{x} \mu_{\tilde{A}}(x) \) is essentially attained.

Let \( \tilde{A}_{i} \in \mathcal{F}(X) \). Then the union of fuzzy sets \( \tilde{A}_{i} \), denoted \( \bigcup \tilde{A}_{i} \), is defined by \( \mu_{\bigcup \tilde{A}_{i}}(x) = \sup_{x} \mu_{\tilde{A}_{i}}(x) = V_{x} \mu_{\tilde{A}_{i}}(x) \), the intersection of fuzzy sets \( \tilde{A}_{i} \), denoted \( \bigcap \tilde{A}_{i} \), is defined by \( \mu_{\bigcap \tilde{A}_{i}}(x) = \inf_{x} \mu_{\tilde{A}_{i}}(x) = A_{x} \mu_{\tilde{A}_{i}}(x) \), and the complement of \( \tilde{A}_{i} \), denoted \( \tilde{A}_{i}^{c} \), is defined by \( \mu_{\tilde{A}_{i}^{c}}(x) = \mu_{\tilde{A}_{i}}(x) \), for all \( x \) in the universal set \( X \).

A fuzzy number \( \tilde{a} \) is a fuzzy set defined on the set of real numbers \( R^{1} \) characterized by
Definition 3.1. CNFN ̃μ is a fuzzy set ̃μ of the real line, such that core of ̃μ is empty and
\[ r^+ ̃μ = \begin{cases} \{ u : ̃μ(\mu) \geq \gamma \} & \text{if } \gamma \in (0, \alpha_{max}^+), \\ \{ u : \gamma \} & \text{if } \gamma = 0 \end{cases} \]
is compact. We use ̃F^*_N for the fuzzy power set of CNFNs.

Definition 3.2. For CNFNs ̃μ and ̃λ with membership functions ̃μ(μ, μ_1) and ̃λ(λ, λ_1), we call ̃Z = ̃μ ⊕ i ̃λ a BCNNFN with membership function μ(̃Z, x) = μ(̃μ, μ_1) ∧ μ(̃λ, λ_1). We denote the class of all the BCNNFNs by ̃F^*_N.

Definition 3.3. For BCNNFNs ̃Z = ̃μ ⊕ i ̃λ, ̃W = ̃γ ⊕ i ̃β, and γ ∈ I^i_0 := [0, α^max_γ] we define ̃μ(γ) = γ and only if (1) ̃μ is upper semicontinuous, (2) μ_2(x) = 0 outside some interval [c, d], (3) There are real numbers a, b such that c ≤ a ≤ b ≤ d and μ_2(x) is increasing on [c, a], μ_2(x) is decreasing on [b, d], μ_2(x) = 1, a ≤ x ≤ b. We denote the set of all fuzzy numbers by ̃F^*. ̃Z is a fuzzy complex number if and only if (1) μ_2(x) is continuous; (2) ̃μ(0, 0) = 0, is open, bounded, connected and simply connected; and (3) ̃Z is non-empty, compact, arcwise connected, and simply connected. We use ̃F^** to the set of all the fuzzy complex numbers.

Let f(z') = f(z') = w be any mapping from C × C into C. Buckley extend f to ̃F^** × ̃F^** into ̃F^** and write f(̃Z, ̃Z') = ̃W if μ_2(w) = V_f(z', z''), μ_2(w) = μ_2(z') ∧ μ_2(z''), respectively. Buckley in [3] proved that ̃Z ⊕ ̃Z' ∈ ̃F^** for any fuzzy complex numbers.

3. Basic definitions and properties

In this section, the concepts of BCNNFNs, GVBCCCNFNs, and other related objects are introduced and some characterizations are presented. The properties of extended operations have been investigated.

Definition 3.4. For BCNNFNs ̃Z = ̃μ ⊕ i ̃λ, ̃W = ̃γ ⊕ i ̃β, and γ ∈ I^i_0 we have ̃W = ̃μ ⊕ i ̃λ if and only if (1) μ_2(x) = 0 outside some interval [c, d], (3) There are real numbers a, b such that c ≤ a ≤ b ≤ d and μ_2(x) is increasing on [c, a], μ_2(x) is decreasing on [b, d], μ_2(x) = 1, a ≤ x ≤ b. We denote the set of all fuzzy numbers by ̃F^*.

Theorem 3.1. For every ̃Z, ̃W ∈ ̃F^*_N and ̃W, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.2. For every ̃Z, ̃W ∈ ̃F^*_N, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.3. For every ̃Z, ̃W ∈ ̃F^*_N, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.4. For BCNNFNs ̃Z = ̃μ ⊕ i ̃λ, ̃W = ̃γ ⊕ i ̃β, and γ ∈ I^i_0 we have
\[ r^+ ̃Z = r^+ ̃μ × r^+ ̃λ, ̃W = (r^+ ̃μ) × (r^+ ̃λ). \]

Theorem 3.5. For every ̃Z, ̃W ∈ ̃F^*_N and ̃W, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.6. For every ̃Z, ̃W ∈ ̃F^*_N, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.7. For every ̃Z, ̃W ∈ ̃F^*_N, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.

Theorem 3.8. For every ̃Z, ̃W ∈ ̃F^*_N, we say that ̃μ ⊕ i ̃λ = ̃W if ̃μ(̃W, z) = μ(̃W, z = μ + i ̃λ) : w = z'' = (μ^2 + λ^2)^0.5.
Theorem 3.11. Let $\tilde{Z}$ be a BCCCNFN and $f$ be an unary operation from complex field $\mathbb{C}$ into $\mathbb{C}$. Then $\mu(f(\tilde{Z}), w) = \Lambda_{w=f(z)}(\mu(\tilde{Z}, z))$.

Proof: We have

$$\mu\left(\left(f\left(N \tilde{Z}\right)\right)_{w}\right) = 1 - \mu(f(N \tilde{Z}), w) = 1 - V_{w=f(z)}(N \tilde{Z}, z) = 1 - \Lambda_{w=f(z)}\left(1 - \mu(\tilde{Z}, z)\right) = 1 - (1 - \Lambda_{w=f(z)}(\mu(\tilde{Z}, z))) = \Lambda_{w=f(z)}(\mu(\tilde{Z}, z))$$

Theorem 3.12. Let $\tilde{Z}, \tilde{W} \in \mathbb{F}_{\mathbb{C}}^{*}$ and $f(z', z'') = w$ be any mapping from $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$, then $\mu(\tilde{Z} \circ \tilde{W}, w) = \Lambda_{w=f(z') \circ f(z'')}(\mu(\tilde{Z}, z') \land \mu(\tilde{W}, z''))$.

Proof: Suppose that $\mu(N \tilde{Z} \circ N \tilde{W}, w)$ attains its value at $(z'_0, z''_0)$. That is, $\mu(N \tilde{Z} \circ N \tilde{W}, w) = V_{f(z'_0, z''_0)=w}(\mu(N \tilde{Z}, z'_0) \land \mu(N \tilde{W}, z''_0)) = \mu(N \tilde{Z}, z'_0) \land \mu(N \tilde{W}, z''_0)$

If $\mu(N \tilde{Z}, z'_0) \land \mu(N \tilde{W}, z''_0) = \mu(N \tilde{W}, z''_0)$ then $\mu(N \tilde{W}, z''_0) \leq \mu(N \tilde{Z}, z'_0)$ and for each $(z', z'')$ such that $f(z', z'') = w$, we have $\mu(N \tilde{W}, z''_0) \geq \mu(N \tilde{Z}, z'_0) \land \mu(N \tilde{W}, z'')$. $\mu(N \tilde{W}, z''_0)$ and for each $(z', z'')$ such that $f(z', z'') = w$, we have $1 - \mu(N \tilde{W}, z''_0) \leq 1 - \mu(N \tilde{Z}, z') \lor 1 - \mu(N \tilde{W}, z'')$.

Second, $\mu(\tilde{Z} \circ \tilde{W}', w) = \mu(\tilde{W}', w) = \mu(\tilde{W}, w)$. $\mu(\tilde{W}', w) = \mu(\tilde{W}, w)' = \mu(\tilde{W}, w)$.
Theorem 3.15. We call $\mu_{[\beta]}(\delta) : R^1 \rightarrow \{[y^-, y^+]; y^- \leq y^+ \text{ and } y^-, y^+ \in I^1[0]\}$ as a

generalized CNFNs (GCNFNs) if $\mu_{[\beta]}(\delta) = [\mu_{\mu}^t(\delta), \mu_{\mu}^u(\delta)]$ for $\mu_{\mu}^t, \mu_{\mu}^u \in \mathbb{F}_{+,\infty}$. The set of all

generalized CNFNs is denoted by $[\mathbb{F}_{+,\infty}]$. We call

$$\lambda([Z], \delta + i \Delta) : C \rightarrow ([y^-, y^+]; y^+ \in [0, \delta_{\max}^Z])$$

is a GRVBCCNFNs if

$$\lambda([Z], \delta + i \Delta) = \lambda([Z], \delta + i \Delta), \lambda([Z], \delta + i \Delta)$$

for BCCCNFNs $Z$. Sometimes, we write $[Z]$ to be $[Z] = [\bar{Z}] = [\bar{Z}]$ and $[Z] = [\bar{Z}] = [\bar{Z}]$.

Definition 3.16. Let $[\bullet] \in \{[+], [-], [\cdot], [\wedge], [\vee]\}$. For GRVBCCNFNs $[\bar{Z}, \bar{Z}], [\bar{W}, \bar{W}]$.

we define $[\bar{Z}, \bar{Z}] [\bullet] [\bar{W}, \bar{W}]$ as follows

$$\lambda([\bar{Z}, \bar{Z}] [\bullet] [\bar{W}, \bar{W}], y + i \beta) =$$

$$\lambda([\bar{Z}, \bar{Z}], [\bullet, \bullet], [\bar{W}, \bar{W}], \mu + i \nu)$$

Theorem 3.17. Let $[Z], [\bar{W}], [\bar{V}] \in [\mathbb{F}_{+,\infty}]$, then

$$[Z] [\bullet] [\bar{W}] \in [\mathbb{F}_{+,\infty}]$$

Proof: The proofs of (2), (3), (4), and (5) are similar to (1), so we only prove (1).

$$[Z, \bar{Z}] [\bullet] ([\bar{W}, \bar{W}] [\bullet] [\bar{V}, \bar{V}])$$

$$= ([Z, \bar{Z}] [\bullet] ([\bar{W}, \bar{W}] [\bullet] [\bar{V}, \bar{V}])$$

$$= ([Z, \bar{Z}] [\bullet] ([\bar{W}, \bar{V}] [\bullet] [\bar{W}, \bar{V}])$$

$$= ([Z, \bar{Z}] [\bullet] ([\bar{W}, \bar{V}] [\bullet] [\bar{W}, \bar{V}])$$
$$= \left(\overline{Z \circ \overline{W}} \right) \left(\overline{(Z \circ \overline{W})}, \overline{(Z \circ \overline{W})}\right)$$

$$= \left(\overline{(Z \circ \overline{W})} \right) \left(\overline{(Z \circ \overline{W})}\right)$$

**Definition 3.19.** Let $[Z] \in [\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]$ and $(r^-, r^+ \in [0, 1]$. We define $[r^-, r^+]$-level of $[Z]$ as $[r^-, r^+] [Z] = r^+ Z \cap r^- Z$.

**Theorem 3.20.** For GRBCCCNFNs $[Z]$ and $[\overline{W}]$ and $[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]$ have $[r^-, r^+] ([Z] [\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]) = [r^-, r^+] [Z] \in [r^-, r^+] [\overline{W}]$.

**Proof:**

$$\lambda([Z] \circ [\overline{W}], r + i \beta) = \lambda(\overline{Z \circ \overline{W}}, r + i \beta), \lambda(\overline{Z \circ \overline{W}}, r + i \beta),$$

$$[r^-, r^+] ([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}])$$

$$= (r^+ \overline{Z \circ \overline{W}} \cap r^- \overline{Z \circ \overline{W}})$$

$$= (r^+ \overline{Z \circ \overline{W}} \cap r^- \overline{Z \circ \overline{W}})$$

$$= (r^+ \overline{Z \circ \overline{W}} \cap r^- \overline{Z \circ \overline{W}})$$

$$= (r^+ \overline{Z \circ \overline{W}} \cap r^- \overline{Z \circ \overline{W}})$$

$$= (r^+ \overline{Z \circ \overline{W}} \cap r^- \overline{Z \circ \overline{W}})$$

$\lambda([Z] \circ [\overline{W}], r + i \beta) = \lambda(\overline{Z \circ \overline{W}}, r + i \beta), \lambda(\overline{Z \circ \overline{W}}, r + i \beta), [r^-, r^+] ([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]).$

**Definition 3.21.** We define a fuzzy distance $f$ of GRBCCCNFNs $[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]$ as follows:

$$f([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]) = \left[\xi(\overline{\alpha}, \overline{\beta}) \vee \xi(\overline{\alpha}, \overline{\beta}), \xi(\overline{\alpha}, \overline{\beta}) \vee \xi(\overline{\alpha}, \overline{\beta})\right],$$

where

$$\xi(\overline{\alpha}, \overline{\beta}) = \left[\left[\overline{\alpha} \vee \overline{\beta}ight] \vee \left[\overline{\alpha} \vee \overline{\beta}\right]\right],$$

and

$$\mathcal{V}[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}] = \left[\mathcal{V}[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}] \vee \mathcal{V}[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]\right].$$

**Theorem 3.22.** For GRBCCCNFNs $[\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]$ we have $f([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]) = f([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}])$. $f([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}]) = f([\mathbb{F}^*_{\mathcal{C}, \mathcal{F}}])$.
\[ f(F([Z]), F([\bar{W}])) \neq f(F([Z]), F([\bar{W}])) + \xi \langle [Z], F([\bar{W}]) \rangle \]

\[ f(F([Z]), F([\bar{W}])) \neq f(F([Z]), F([\bar{W}])) + \xi \langle [Z], F([\bar{W}]) \rangle \]

\[ f(F([Z]), F([\bar{W}])) \neq f(F([Z]), F([\bar{W}])) + \xi \langle [Z], F([\bar{W}]) \rangle \]

\[ f(F([Z]), F([\bar{W}])) \neq f(F([Z]), F([\bar{W}])) + \xi \langle [Z], F([\bar{W}]) \rangle \]

\[ f(F([Z]), F([\bar{W}])) \neq f(F([Z]), F([\bar{W}])) + \xi \langle [Z], F([\bar{W}]) \rangle \]

Theorem 4.4. Let \( \hat{f} \) and \( \hat{g} \) are continuous at GRVBFCCNFs \( \tilde{[Z]} \) and \( \tilde{F}(\tilde{[Z]}) \) respectively, then \( \hat{g}(\hat{f}(\tilde{[Z]})) \) is continuous at \( \tilde{[Z]} \).

Proof: By hypothesis, for \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \) when \( \hat{g}(\hat{f}(\tilde{[Z]}), \tilde{F}(\tilde{[Z]})) < \delta \), and for \( \delta > 0 \), there exists \( \delta' > 0 \), such that when \( \hat{f}(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \delta' \), \( \hat{g}(\hat{f}(\tilde{[Z]}), \tilde{F}(\tilde{[Z]})) < \delta \). Therefore, for any \( \epsilon > 0 \), we seek find \( \delta' > 0 \) such that \( f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \) when \( \hat{g}(\hat{f}(\tilde{[Z]}), \tilde{F}(\tilde{[Z]})) < \delta' \).

Theorem 4.5. \( \hat{f} \) is continued at GRVBFCCNFs \( \tilde{[Z]} \) if and only if the real part \( \text{Re}\hat{f}(\tilde{[Z]}) \) and imaginary part \( \text{Im}\hat{f}(\tilde{[Z]}) \) are continued at \( \tilde{[Z]} \).

Proof: For \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \) when \( \hat{f}(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \delta \). And

\[ f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \]

\[ f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \]

\[ f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \]

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\[ f(\tilde{[Z]}, \tilde{F}(\tilde{[Z]})) < \epsilon \]

This implies that

\[ \xi((\text{Re}\hat{f}(\tilde{[Z]}))^\epsilon, (\text{Re}\hat{f}(\tilde{[Z]}))^\epsilon) < \epsilon, \]

\[ \xi((\text{Im}\hat{f}(\tilde{[Z]}))^\epsilon, (\text{Re}\hat{f}(\tilde{[Z]}))^\epsilon) < \epsilon, \]

\[ \xi((\text{Re}\hat{f}(\tilde{[Z]}))^\epsilon, (\text{Im}\hat{f}(\tilde{[Z]}))^\epsilon) < \epsilon, \]

\[ \xi((\text{Im}\hat{f}(\tilde{[Z]}))^\epsilon, (\text{Im}\hat{f}(\tilde{[Z]}))^\epsilon) < \epsilon. \]

Hence

\[ \pi(\text{Re}\hat{f}(\tilde{[Z]}), \text{Re}\hat{f}(\tilde{[Z]})) < \epsilon \text{ and} \]

\[ \pi(\text{Im}\hat{f}(\tilde{[Z]}), \text{Im}\hat{f}(\tilde{[Z]})) < \epsilon. \]

Theorem 4.6. Let \( \hat{f} \) is continuous at \( \tilde{[W]} \in \tilde{[F]} \), then there exists GRVBFCCNFs \( [Z_0], [\tilde{Z}_1] \) and \( \delta > 0 \) satisfy \( [Z_1] \neq \hat{f}(\tilde{[Z]}) < [Z_0] \) when \( \hat{f}([Z]), \tilde{[W]} \rightarrow [Z_0] \). So that, for any \( \gamma \in [1, \gamma] \)

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon \]

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon \]

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon \]

Hence

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon, \]

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon, \]

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon, \]

and

\[ \gamma^\epsilon(\text{Re}\hat{f}([\tilde{W}])) < \epsilon. \]
Hence, there exists \([\tilde{\alpha}_0], [\tilde{\alpha}_1], [\tilde{\alpha}_0], [\tilde{\alpha}_1] \in [\mathcal{F}_{\alpha, \gamma}]\] such that \([\tilde{\alpha}_0] < \Re \tilde{F}(\{\tilde{Z}\}) < [\tilde{\alpha}_0]\) and \([\tilde{\alpha}_1] < \Im \tilde{F}(\{\tilde{Z}\}) < [\tilde{\alpha}_1]\). This completes the proof by taking \(\tilde{Z}_0 = [\tilde{\alpha}_0] + i[\tilde{\alpha}_1]\) and \([\tilde{Z}_1] = [\tilde{\alpha}_0] + [\tilde{\alpha}_1]\).

**Theorem 4.7.** Let \(\tilde{F}\) is continuous at \([\tilde{W}] \in [\mathcal{F}_{\alpha, \gamma}]\), \([Z_0] \in [\mathcal{F}_{\alpha, \gamma}]\), and there exists \(\delta > 0\) such that \(\tilde{F}(\{\tilde{Z}\}) < [Z_0]\) (resp. \(\tilde{F}(\{\tilde{Z}\}) > [Z_0]\)) when \(\tilde{F}(\{\tilde{Z}\}, [\tilde{W}]) < \delta\) then \(\tilde{F}(\{\tilde{W}\}) < [Z_0]\) (resp. \(\tilde{F}(\{\tilde{W}\}) > [Z_0]\)).

**Proof:** From Theorem 4.6, we have

\[
\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^e < \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^e + \epsilon
\]

(resp. \(\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^e > \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^e - \epsilon\))

\[
\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^u < \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^u + \epsilon
\]

(resp. \(\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^u > \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^u - \epsilon\))

\[
\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^u < \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^u + \epsilon
\]

(resp. \(\gamma^+ (\Re \tilde{F}(\{\tilde{W}\})^u > \gamma^+ (\Re \tilde{F}(\{\tilde{Z}\})^u - \epsilon\))

Taking

\[
\epsilon = \gamma^+ (\Re [\tilde{Z}_0])^e - \gamma^+ (\Re [\tilde{Z}_1])^e
\]

(resp. \(\epsilon = \gamma^+ (\Re [\tilde{Z}_0])^e - \gamma^+ (\Re [\tilde{Z}_0])^e\))

\[
\epsilon = \gamma^+ (\Re [\tilde{Z}_0])^u - \gamma^+ (\Re [\tilde{Z}_1])^u
\]

(resp. \(\epsilon = \gamma^+ (\Re [\tilde{Z}_1])^u - \gamma^+ (\Re [\tilde{Z}_0])^u\)).
Letting ‘$M, M', M''$, and $M'''$ denotes $\Delta^{+}(\text{Re}[Z])$, $\Delta^{+}(\text{Re}[Z])$, $\Delta^{+}(\text{Im}[Z])$, and $\Delta^{+}(\text{Im}[Z])$, respectively, and $M = 'M' \cup 'M'' \cup 'M'''$ Since $\bar{F}$ is continued at $[\bar{W}]$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\bar{F}([Z], \bar{F}([\bar{W}])) < \varepsilon / M$ as $F([Z], \bar{F}([\bar{W}])) < \delta_0$. 

Let $\delta_0 = \delta_1 \wedge \delta_2 \wedge \delta_3$. Then we have

\[
\bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0 \Rightarrow \bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0
\]

We first calculate

\[
\bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0 \Rightarrow \bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0
\]

Similarly can be shown that

\[
\bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0 \Rightarrow \bar{F}([Z], \bar{F}([\bar{W}])) < \delta_0
\]

Hence
\[ F\left(\tilde{F}(\tilde{Z}_o)[\cdot]\tilde{F}(\tilde{Z}), \tilde{F}(\tilde{Z}_o)[\cdot]\tilde{F}(\tilde{W})\right) \leq M, \tilde{F}\left(\tilde{F}(\tilde{Z}), \tilde{F}(\tilde{W})\right) \leq M, E/M = \tilde{E} \quad \text{as} \quad \tilde{F}(\tilde{Z}, \tilde{W}) < \tilde{\delta}^* \text{. That is } \tilde{F}(\tilde{Z}_o)[\cdot]\tilde{F}(\tilde{Z}) \text{ is fuzzy continuous at } \tilde{W}. \]

5. Fuzzy Complex Derivatives

In this section, we will use the “dot” notation for partial derivatives with respect to \( z \). Otherwise, we employ the “prime” notation for the derivative of a complex function of one variable. Furthermore, we use the standard notations and results of Yang and Yi in [24].

The complex fuzzy valued function \( \tilde{f} : \mathbb{C} \rightarrow \tilde{\mathbb{F}}^{*}_{\mathbb{R}} \) is fuzzy differentiable in its domain if the derivative of \( \gamma \tilde{f}(\tilde{z}) = z_{y^{y^{y^{f}}}} \tilde{f} \) denoted by \( z_{y^{y^{y^{f}}}^{y^{f}}} \tilde{f} \) exists for all \( \gamma \in I^1_{\mathbb{R}} \). We call \( \tilde{f} \) is fuzzy meromorphic if \( z_{y^{y^{y^{f}}}^{y^{f}}} \tilde{f} \) is meromorphic for any \( \gamma \in I^1_{\mathbb{R}} \).

We say that \( \tilde{f} \) has a pole (resp. zero) if \( z_{y^{y^{y^{f}}}^{y^{f}}} \) has a poles (resp. zeros) for any \( \gamma \in I^1_{\mathbb{R}} \).

**Theorem 5.1.** Let \( \tilde{Z}, \tilde{W} \in \tilde{\mathbb{F}}^{*}_{\mathbb{R}} \) and \( \tilde{f} \) be a fuzzy meromorphic function. If \( \tilde{f}(\tilde{z}) = \tilde{W} \) and \( \tilde{f}'(\tilde{z}) = \tilde{W} \) have the same zeros with the same order, \( \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) = o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \) then \( \tilde{f}' = \tilde{f} \).

**Proof:** Assume that \( z_{y^{y^{y^{f}}}^{y^{f}}} \neq z_{y^{y^{y^{f}}}^{y^{f}}}. \)

\[ 2T(r, z_{y^{y^{y^{f}}}^{y^{f}}}) \leq \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ = \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ \leq \tilde{T}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ \leq \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ = o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})). \]

**Theorem 5.2.** Let \( \tilde{Z}, \tilde{W} \in \tilde{\mathbb{F}}^{*}_{\mathbb{R}} \) and \( \tilde{f} \) be a fuzzy meromorphic function. If \( \tilde{f}(\tilde{z}) = \tilde{W} \) and \( \tilde{f}'(\tilde{z}) = \tilde{W} \) have the same zeros with the same order, \( \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) = o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \) then \( \tilde{f}' = \tilde{f} \).

**Proof:** Assume that \( z_{y^{y^{y^{f}}}^{y^{f}}} \neq z_{y^{y^{y^{f}}}^{y^{f}}}. \)

\[ 2T(r, z_{y^{y^{y^{f}}}^{y^{f}}}) \leq \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ = \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ \leq \tilde{T}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ \leq \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + \tilde{N}(r, z_{y^{y^{y^{f}}}^{y^{f}}}) + o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})) \]

\[ = o(T(r, z_{y^{y^{y^{f}}}^{y^{f}}})). \]
\[
\begin{align*}
\leq m\left(r, \frac{1}{\mathcal{N}}\right) + 2\mathcal{N}(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}})) \\
\leq T\left(r, \frac{1}{\mathcal{N}}\right) + 2\mathcal{N}(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}})) \\
\leq T\left(r, z_{y+\hat{f}}\right) + 2\mathcal{N}(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}})) \\
\leq T\left(r, z_{y+\hat{f}}\right) + \mathcal{N}(r, z_{y+\hat{f}}) + 2\mathcal{N}(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}})) \\
\leq 2T\left(r, z_{y+\hat{f}}\right) + T(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}})) \\
\end{align*}
\]

Hence
\[
2T\left(r, z_{y+\hat{f}}\right) + 2T(r, z_{y+\hat{f}}) \leq 2T\left(r, z_{y+\hat{f}}\right) + T(r, z_{y+\hat{f}}) + o(T(r, z_{y+\hat{f}}))
\]

This implies that \(T(r, z_{y+\hat{f}}) \leq o(T(r, z_{y+\hat{f}}))\).

**Open question:** What happens if we consider functions mapping subset of GRVBCCCNFNs into \([\mathcal{F}^{\ast}_{\infty}]\) to define the fuzzy derivative as follows:

Let \([\mathcal{Z}], [\hat{\mathcal{Z}}] \in [\mathcal{F}^{\ast}_{\infty}]\). We say the derivative \(D\mathcal{F}(\mathcal{Z})\) exists if there exists \([\mathcal{W}] \in [\mathcal{F}^{\ast}_{\infty}]\) such that \((\mathcal{F}(\mathcal{Z})[-\hat{\mathcal{F}}(\mathcal{Z})])/(\mathcal{Z}[-\hat{\mathcal{Z}}]))\) \(\mathcal{F}\)-converges to \([\mathcal{W}]\) as \([\mathcal{Z}] \mathcal{F}\)-converges to \([\hat{\mathcal{Z}}]\).

**References**


